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A Statistical Problem Related to the Launching of a Missile

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CONTENTS

ABSTRACT

A statistical model of the delays encountered in the countdown of a missile launch is formulated. Both exact and approximate results are derived for the probability distribution of the number of days elapsed until two firings are accomplished. For a specific set of parameter values, the approximate and exact results are compared.

I. INTRODUCTION

The launch date of a missile or an interplanetary probe is usually very critical with regard to payload or energy expenditure. If several launches are to be attempted from the same pad in a given period of time, the launch dates must be chosen with due regard to the relatively fixed delay time necessary to clean the pad after the first firing and the random delays due to failure to complete the countdown procedures. It is with the latter type delay that this Report is concerned.

In Sec. II, a mathematical model is formulated and the principal results are given. In Sec. III, the exact results for a specific set of parameter values are compared with the approximate results developed in Sec. II. The principal formulae are derived in Sec. IV.

II. THE PROBLEM AND PRINCIPAL RESULTS

The countdown to launch a missile is completed with a probability p_0. If the countdown is completed, another launch is attempted the following day. If the countdown fails to be completed, there is a delay of k days ($k = 1, 2, 3, 4$) before another can be attempted. The probability of a k-day delay is p_k. Each launch is independent of the previous attempts. The process terminates as soon as two countdowns are completed.

If n is the number of attempted countdowns, and N is the number of days until two completed countdowns, then the probability densities of these functions are given by

$$P_1(n) = (n-1)\, p_0^2 (1-p_0)^{-2} \qquad n = 2, 3, \cdots \tag{1}$$

$$P_2(N) = p_0^2 \sum_N \frac{(n_1+n_2+n_3+n_4+1)!}{n_1!\,n_2!\,n_3!\,n_4!} p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$$
$$\sum k\, n_k + 2 = N \qquad N = 2, 3, \cdots \tag{2}$$

respectively. The moment-generating function for each of these distributions is given by

$$M_1(\theta) = E[e^{\theta n}] = \frac{p_0^2 e^{2\theta}}{(1-q_0 e^{\theta})^2} \tag{3}$$

$$M_2(\theta) = E[e^{\theta N}] = \frac{p_0^2 e^{2\theta}}{[1-(p_1 e^{\theta} + p_2 e^{2\theta} + p_3 e^{3\theta} + p_4 e^{4\theta})]^2} \tag{4}$$

respectively, where $q_0 = 1 - p_0$ and $E[e^{\theta n}]$ means the expected value of $e^{\theta n}$

$$E[e^{\theta n}] = \sum_n e^{\theta n} P_1(n)$$

The mean values and variances of these variables are given by

$$\mu_n = \frac{2}{p_0} \qquad \sigma_n^2 = \frac{2(1-p_0)}{p_0^2} = \frac{2q_0}{p_0^2} \tag{5}$$

$$\mu_N = 2 + \frac{2\sum_1^4 k p_k}{p_0} \qquad \sigma_N^2 = \frac{2(\sum_1^4 k p_k)^2}{p_0^2} + \frac{2\sum_1^4 k^2 p_k}{p_0} \tag{6}$$

These formulae will be useful when approximating these distributions by normal distributions.

The $P_2(N)$ can be evaluated directly from Eq. (2). They can also be generated by the function

$$M(\xi) = \frac{p_0^2 \xi^2}{[1-(p_1\xi + p_2\xi^2 + p_3\xi^3 + p_4\xi^4)]^2} = \sum_{N=2}^{\infty} P_2(N)\, \xi^N \tag{7}$$

They can also be obtained from the recursion relation

$$P_2(N+3) = p_1 \frac{N+2}{N+1} P_2(N+2) + p_2 \frac{N+3}{N+1} P_2(N+1) + p_3 \frac{N+4}{N+1} P_2(N) + p_4 \frac{N+5}{N+1} P_2(N-1) \tag{8}$$

with

$$P_2(2) = p_0^2, \quad P_2(1) = P_2(0) = P_2(-1) = 0 \tag{9}$$

Using any of the above techniques, it is found, for example, that

$$\begin{aligned} P_2(2) &= p_0^2 \\ P_2(3) &= 2 p_0^2 p_1 \\ P_2(4) &= p_0^2 (3 p_1^2 + 2 p_2) \\ P_2(5) &= 2 p_0^2 (2 p_1^3 + 3 p_1 p_2 + p_3) \end{aligned} \tag{10}$$

et cetera.

If the probability density function is approximated by that of the normal distribution by matching means and standard deviations, then the following results may be obtained:

$$\begin{aligned} Pr\{N \leq N_0\} = k &\approx Pr\{\sigma_N t + \mu_N \leq N_0 + 0.5\} = k \\ &= Pr\left\{ t \leq \frac{N_0 + 0.5 - \mu_N}{\sigma_N} \right\} = k \\ &= Pr\{t \leq t_0\} = k \end{aligned} \tag{11}$$

where t is a standard normal variable (normally distributed with mean zero and standard deviation one), μ_N and σ_N are given by Eq. (6) and

$$t_0 = \frac{N_0 + 0.5 - \mu_N}{\sigma_N}$$
$$N_0 = [\sigma_N t_0 + \mu_N - 0.5] + 1 \tag{12}$$

where [ ] means the greatest integer in N_0. The following tabulation is useful:

k	0.90	0.95	0.99	0.999
t_0	1.28	1.65	2.33	3.09

III. A SPECIFIC EXAMPLE

Suppose that, for a given p_0,

$$p_1 = 0.4\,(1 - p_0)$$
$$p_2 = 0.3\,(1 - p_0)$$
$$p_3 = 0.2\,(1 - p_0)$$
$$p_4 = 0.1\,(1 - p_0) \qquad (13)$$

so that

$$\mu_N = \frac{4 - 2p_0}{p_0}, \qquad \sigma_N^2 = \frac{2\,(1 - p_0)\,(4 + p_0)}{p_0^2} \qquad (14)$$

and we obtain the following:

p_0	μ_n	σ_n	N	σ_N
0.1	20	13.42	38	27.17
0.2	10	6.32	18	12.96
0.3	6.7	3.94	11.33	8.18
0.4	5.0	2.74	8	5.75
0.5	4.0	2.00	6	4.24
0.6	3.3	1.49	4.67	3.20
0.7	2.9	1.11	3.71	2.40
0.8	2.5	0.79	3	1.73
0.9	2.2	0.50	2.44	1.10

For this example, the exact distribution was calculated using the recursion relation Eq. (8) and compared with the approximate results obtained from Eq. (12). The results are given below.

p_0	90% Level		95% Level		99% Level	
	N_0 (approx.)	N_0 (exact)	N_0 (approx.)	N_0 (exact)	N_0 (approx.)	N_0 (exact)
0.1	73	*	83	*	101	*
0.2	35	*	39	*	48	*
0.3	22	22	25	27	30	*
0.4	15	16	17	19	21	*
0.5	11	12	13	14	16	20
0.6	9	9	10	11	12	16
0.7	7	7	8	9	9	12
0.8	5	5	6	6	7	9
0.9	4	4	4	5	5	7

*These points were not calculated.

The preceding results show a remarkably good comparison between the approximate and exact results despite the fact that the distributions are far from normal. Note that the approximation is poorer at the higher confidence levels ($k = 99\%$). This is due to the fact that the probability density does not tend to zero as rapidly as does the normal. It is possible to improve these estimates but it is doubtful that it would be worthwhile.

IV. DERIVATIONS

Equation (1) is simply the generalized Pascal distribution. Briefly, to obtain two successful countdowns in exactly n days, there must be one success and $(n-2)$ failures in $(n-1)$ days followed by one success on the nth day. The number of ways in which the former can be done is $\binom{n-1}{1}$, each having a probability $p_0(1-p_0)^{n-2}$; hence, the result.

Equation (2) is obtained as follows: If there are n_k delays of the type leading to a k-day delay $(k = 1, 2, 3, 4)$ and one success, these can be permuted

$$\frac{(n_1+n_2+n_3+n_4+1)!}{1!\,n_1!\,n_2!\,n_3!\,n_4!}$$

different ways, each occurring with probability

$$p_0\, p_1^{n_1}\, p_2^{n_2}\, p_3^{n_3}\, p_4^{n_4}$$

If this is followed by a second success, the probability is then

$$\frac{(n_1+n_2+n_3+n_4+1)!}{1!\,n_1!\,n_2!\,n_3!\,n_4!}\, p_0^2\, p_1^{n_1}\, p_2^{n_2}\, p_3^{n_3}\, p_4^{n_4}$$

If this is summed over all partitions of N such that $\sum k n_k + 2 = N$, this will be the desired probability, since the sum represents the number of different ways in which we can have exactly N days elapsed, times their respective probabilities.

Equation (3) is obtained as follows:

$$\begin{aligned} M_1(\theta) &= \sum_2^\infty (n-1)\, p_0^2 (1-p_0)^{n-2} e^{n\theta} \\ &= p_0^2 e^\theta \frac{d}{d\theta} \sum_2^\infty (1-p_0)^{n-2} e^{(n-1)\theta} \\ &= p_0^2 e^\theta \frac{d}{d\theta}\left[\frac{e^\theta}{1-(1-p_0)e^\theta}\right] = p_0^2 e^\theta \frac{d}{d\theta}\frac{1}{e^{-\theta}-(1-p_0)} \\ &= p_0^2 e^\theta \frac{e^{-\theta}}{[e^{-\theta}-(1-p_0)]^2} \\ &= \frac{p_0^2}{[e^{-\theta}-(1-p_0)]^2} = \frac{p_0^2 e^{2\theta}}{[1-(1-p_0)e^\theta]^2} \\ &= \frac{p_0^2 e^{2\theta}}{(1-q_0 e^\theta)^2} \end{aligned}$$

Equation (4) can be obtained in a similar manner. Another approach which is reproduced here illustrates another attack on the problem. The joint density function of the n_k and n is given by

$$f(n_1, n_2, n_3, n_4; n) = \frac{(n-1)!}{1!\,n_1!\,n_2!\,n_3!\,n_4!}\, p_0^2\, p_1^{n_1}\, p_2^{n_2}\, p_3^{n_3}\, p_4^{n_4}$$

$$\text{and} \sum n_k = n-2, \quad n = 2, 3, \cdots \tag{15}$$

The joint moment-generating function of the n_k, n is given by

$$\begin{aligned} M(\theta_1,\theta_2,\theta_3,\theta_4;\alpha) &= \sum_{n=2}^\infty \sum_{\Sigma n_k = n-2} \frac{(n-1)!\, e^{\alpha n} p_0^2}{n_1!\,n_2!\,n_3!\,n_4!} \\ &\quad \times (p_1 e^{\theta_1})^{n_1} (p_2 e^{\theta_2})^{n_2} (p_3 e^{\theta_3})^{n_3} (p_4 e^{\theta_4})^{n_4} \\ &= \sum_{n=2}^\infty p_0^2 e^{\alpha n} (n-1)(p_1 e^{\theta_1} + p_2 e^{\theta_2} + p_3 e^{\theta_3} + p_4 e^{\theta_4})^{n-2} \\ &= \frac{p_0^2 e^{2\alpha}}{[1-(p_1 e^{\theta_1} + p_2 e^{\theta_2} + p_3 e^{\theta_3} + p_4 e^{\theta_4})e^\alpha]^2} \end{aligned} \tag{16}$$

the latter sum being obtained as above. Note that, if one sets the $\theta_i \equiv 0$, $M_1(\alpha)$ is obtained, since $p_1 + p_2 + p_3 + p_4 = 1 - p_0$. Now the moment-generating function of the random variable $N = \sum k n_k + 2$ is

$$\begin{aligned} E[e^{\theta N}] &= E[e^{\theta n_1 + 2\theta n_2 + 3\theta n_3 + 4\theta n_4 + 2\theta}] \\ &= e^{2\theta} E[e^{\theta n_1 + 2\theta n_2 + 3\theta n_3 + 4\theta n_4}] \end{aligned}$$

which can be obtained from the joint moment-generating function by setting $\alpha = 0$, $\theta_k = k\theta$ so that

$$\begin{aligned} M_2(\theta) &= e^{2\theta} M(\theta, 2\theta, 3\theta, 4\theta; 0) \\ &= \frac{p_0^2 e^{2\theta}}{[1-(p_1 e^\theta + p_2 e^{2\theta} + p_3 e^{3\theta} + p_4 e^{4\theta})]^2} \end{aligned} \tag{17}$$

It is an easy matter to obtain the mean and variance of n and N from their respective moment-generating functions since

$$\mu_n = \left.\frac{dM_1}{d\theta}\right|_{\theta=0} \quad \text{and} \quad \sigma_n^2 = \left.\frac{d^2M_1}{d\theta^2}\right|_{\theta=0} - \mu_n^2$$

with corresponding formulae for μ_N and σ_N^2. The computations are straightforward and yield Eqs. (5) and (6).

Equation (7) is obtained as follows: If in $M_2(\theta)$ θ is replaced by $j\omega$, the characteristic function of the distribution (i.e., the Fourier Transform of the density function) is obtained. If the distribution is discrete, the density function can be represented by

$$f(x) = \sum_n a_n \delta(x-n)$$

whose Fourier Transform becomes

$$\sum_n a_n e^{j\omega n}$$

when $a_n = Pr\{x = n\}$. If we set $\xi = e^{j\omega} = e^{\theta}$ and expand the moment-generating function $M(\xi)$ as a power series in ξ, the coefficient of ξ^n is then $Pr\{x = n\}$, from which Eq. (7) follows.

The Eq. (8) is the application of this concept. The relation itself is obtained in the following way: Note that the coefficient of ξ^n is given by

$$\frac{1}{n!} \left. \frac{d^n M_n(\xi)}{d\xi^n} \right|_{\xi=0}$$

Now, consider the relation

$$M(\xi)\left[1 - \sum_i p_i \xi^i\right]^2 = 1 \tag{18}$$

where the term $p_0 \xi^2$ is ignored for the time being. Differentiating Eq. (18) yields

$$M^{(1)}\left[1 - \sum_i p_i \xi^i\right] - 2M\sum_i i\, p_i \xi^{i-1} = 0 \tag{19}$$

Differentiating n times and using Leibnitz' rule,

$$\sum_{k=0}^{n} \binom{n}{k}\left[1 - \sum_i p_i \xi^i\right]^{(k)} M^{(n-k+1)}(\xi) - 2\sum_{k=0}^{n} \binom{n}{k}\left[\Sigma\, i\, p_i \xi^{i-1}\right]^{(k)} M^{(n-k)}(\xi) = 0 \tag{20}$$

If one expands Eq. (20) and evaluates at $\xi = 0$, there results the equation

$$\begin{aligned} M^{(n+1)}(0) = {} & p_1(n+2)M^{(n)} + p_2\, n(n+3)M^{(n-1)} \\ & + p_3\, n(n-1)(n+4)M^{(n-2)} \\ & + p_4\, n(n-1)(n-2)(n+5)M^{(n-3)} \end{aligned} \tag{21}$$

Now note that

$$P_2(N) = \frac{M^{(N-2)}(0)}{(N-2)!} \tag{22}$$

which accounts for the omitted ξ^2 term in Eq. (18); when this is substituted into Eq. (21), the recursion relation defined by Eq. (8) is obtained. The initial condition clearly accounts for the omitted p_0^2.

The normal approximations to a discrete variable are quite standard and require no additional explanation.